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Block transformations of one-dimensional deterministic cellular automaton rules

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Abstract. Given a one-dimensional cellular automaton rule f, a block transform of f is a rule $T_b f$ such that there exists between the limit sets of both rules a bijection that replaces each site value x in a configuration belonging to the limit set of f with a string $x^b = xx\cdots x$ of length b in the corresponding configuration belonging to the limit set of $T_b f$. If f is totalistic, there exists a unique totalistic block transform and a large number of non-totalistic block transforms $T_b f$. If f is not totalistic, there are no totalistic block transforms but there still exists a large number of non-totalistic block transforms. Their number increases very rapidly with the block size b and the range r of f. The range of $T_b f$ may be any integer greater than or equal to rb. Many block transforms are studied. The evolution according to rule $T_b f$ towards its limit set is discussed in terms of the annihilation of defects. These defects are often simply related to the defects characterizing the evolution according to rule f.

1. Introduction

A cellular automaton (CA) consists of a lattice with a discrete variable at each site. Each site variable evolves in discrete time steps according to a definite rule that involves the values of neighbouring site variables at previous time steps. The site variables are updated simultaneously. CA provide simple models for a variety of complex systems containing large numbers of identical elements with local interactions (Farmer *et al* 1984, Wolfram 1986, Manneville *et al* 1989, Gutowitz 1990).

CA may be considered as discrete dynamical systems. Let $s: \mathbb{Z} \times \mathbb{N} \mapsto \{0,1\}$ be a function that satisfies the equation

$$(\forall i \in \mathbb{Z}) \quad (\forall t \in \mathbb{N}) \qquad s(i,t+1) = f\big(s(i-r,t), s(i-r+1,t), \ldots, s(i+r,t)\big)$$

and such that

$$(\forall i \in \mathbb{Z}) \quad s(i,0) = s_0(i)$$

where $\mathbb{N}$ is the set of non-negative integers, $\mathbb{Z}$ is the set of all integers and $s_0: \mathbb{Z} \to \{0,1\}$ is a given function that specifies the initial condition. Such a discrete dynamical system is a one-dimensional CA. The map $f: \{0,1\}^{2r+1} \to \{0,1\}$ determines the dynamics. It is referred to as the local rule of the CA. The positive integer r is the range of the rule. The function $S_t: i \mapsto s(i,t)$ is the state of the CA at time t. $\mathcal{S} = \{0,1\}^{\mathbb{Z}}$ is the state

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space. An element of the state space is also called a configuration. Since the state at time $t+1$ is entirely determined by the state at time t and the rule f, f induces a mapping $\mathbf{f}: \mathcal{S} \to \mathcal{S}$ such that

$$S_{t+1} = \mathbf{f}(S_t).$$

$\mathbf{f}$ is called the global rule of the CA.

Based on investigations of a large sample of CA, Wolfram (1984a, b) has shown that, according to their asymptotic behaviour, CA rules appear to fall into four qualitative classes. Class-1 CA evolve from almost all initial states to a unique homogeneous state in which all sites have the same value. Class-2 CA yield separated simple stable or periodic structures. Class-3 CA exhibit chaotic patterns. The statistical properties of these patterns are typically the same for almost all initial states. In particular, the density of non-zero site variables tends to a fixed value as time t tends to ∞. The evolution of class-4 CA leads to complex localized or propagating structures.

Consider a rule f; its limit set is defined by

$$\begin{aligned}\Lambda_f &= \lim_{t\to\infty} \mathbf{f}^t(\mathcal{S}) \\ &= \bigcap_{t\geq 0} \mathbf{f}^t(\mathcal{S}).\end{aligned}$$

Λ_f is clearly invariant, i.e. $\mathbf{f}(\Lambda_f) = \Lambda_f$. Since any $\mathbf{f}$-invariant subset belongs to Λ_f, the limit set is the maximal $\mathbf{f}$-invariant subset of $\mathcal{S}$.

This paper concentrates on one aspect of CA theory. It deals with block transformations of one-dimensional CA rules. It is a step towards the solution of a problem stated by Wolfram (1985) concerning the scaling properties of CA. Given a CA rule f, the idea is to replace 0 and 1, in the spatio-temporal pattern generated by f, by two blocks B_0 and B_1, respectively, and search if the resulting pattern could be generated by a local CA rule F said to be a 'block transform' of f.

Here we only consider the case where B_0 and B_1 are two sequences of identical digits of the same length.

In what follows, given a rule f and a positive integer b, we call the 'block transform' of f a rule $T_b f$ such that

$$\Lambda_{T_b f} = H_b\left(\Lambda_f\right) \tag{1}$$

where the mapping $H_b: \mathcal{S} \to \mathcal{S}$ is a homomorphism defined by

$$(\forall \cdots xyz \cdots \in \mathcal{S}) \qquad H_b(\cdots xyz \cdots) = \cdots x^b y^b z^b \cdots.$$

x^b stands for x repeated b times; for instance, $x^3 = xxx$. In practice, we prefer to use the following criterion.

$T_b f$ is a block transform of f if it satisfies the following two conditions.

(i)

$$T_b\mathbf{f} \circ H_b = H_b \circ \mathbf{f}. \tag{2}$$

(ii) The number of sequences of zeros and ones, whose length is not a multiple of b, in any configuration S_t, generated by $T_b f$ after t time steps, tends to zero as t tends to ∞.

Note that it follows from relation (2) that $T_b \mathbf{f}$ leaves $H_b(\Lambda_f)$ invariant, and since a limit set is a maximal invariant subset

$$H_b\left(\Lambda_f\right) \subseteq \Lambda_{T_b f}. \tag{3}$$

The equality results from the second condition about the decrease of the numbers of 'defects' (i.e. sequences of zeros and ones whose lengths are not a multiple of b).

After sufficiently many iterations the spatio-temporal patterns generated by rules f and $T_b f$ cannot be distinguished after a space contraction by a factor b of the latter. This implies that rules f and $T_b f$ belong to the same class. Moreover, for class-3 CA, the asymptotic density of non-zero site variables is invariant under a block transformation. If the size of the lattice is large but finite, say equal to N, the parameter b defines a characteristic length, and the probability distribution of the asymptotic density, as a function of b and N, should be homogeneous, i.e. a function of the ratio b/N.

Formal language theory may be used to obtain more complete characterizations of limit sets (Wolfram 1984b). A language is a set of words formed of letters in a finite alphabet according to definite grammatical rules (Hopcroft and Ullman 1969, Denning *et al* 1978). Four types of formal grammars can be specified. The simplest type is said to be regular. Languages generated by regular grammars may be specified by finite-state machines represented by finite directed graphs (figure 1).

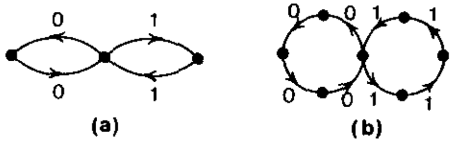

Figure 1. Graphs representing regular grammars. (*a*) Range-1 totalistic rule 6. (*b*) Range-2 totalistic rule 30.

The limit set of a given rule may be regarded as a formal language. The alphabet of terminal letters is $\{0, 1\}$. All words in this alphabet have infinite length. Since all four classes of formal languages are closed under the homomorphism H_b (Hopcroft and Ullman 1969), the languages Λ_f and $\Lambda_{T_b f}$ belong to the same class. In particular, if a limit set Λ_f is a regular language, then the limit set $\Lambda_{T_b f}$ is also regular, and its graph is obtained by replacing each arc by a path consisting of b arcs. If the original arc carries the terminal letter a, all arcs in the path carry the letter a.

2. Totalistic rules

A rule f is said to be totalistic (Wolfram 1984a) if, for all integers i and all positive integers t, there exists a function $f^{\text{tot}}\colon \{0, 1, \ldots, 2r + 1\} \to \{0, 1\}$ such that

$$\begin{aligned} s(i, t+1) &= f\big(s(i-r, t), s(i-r+1, t), \ldots, s(i+r, t)\big) \\ &= f^{\text{tot}}\big(s(i-r, t) + s(i-r+1, t) + \cdots + s(i+r, t)\big). \end{aligned} \tag{4}$$

In this case, the rule f may be specified by a numerical code

$$C_f = \sum_{n=0}^{2r+1} 2^n f^{\mathrm{tot}}(n).$$

For instance, the code of the range-2 rule f defined by

$$f(x_1, x_2, x_3, x_4, x_5) = \begin{cases} 1 & \text{if } 0 < x_1 + x_2 + x_3 + x_4 + x_5 < 5 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

is $C_f = 2 + 2^2 + 2^3 + 2^4 = 30$.

Given a range-r totalistic rule f, for all positive integers b, there exists a unique range-rb totalistic rule $T_b f$, which satisfies the relation (2)

$$T_b\mathbf{f} \circ H_b = H_b \circ \mathbf{f}.$$

Let us first establish this result for $r = 1$ and $b = 2$. In this particular case, in order to satisfy equation (2) given a totalistic rule f we have to determine a totalistic rule $T_2 f$ such that

$$T_2 f(x_1, x_1, x_2, x_2, x_3) = T_2 f(x_1, x_2, x_2, x_3, x_3) = f(x_1, x_2, x_3) \tag{6}$$

for all $(x_1, x_2, x_3) \in \{0, 1\}^3$. Since f and $T_b f$ are totalistic, there exist two functions $f^{\mathrm{tot}}: \{0, 1, 2, 3\} \to \{0, 1\}$ and $T_2 f^{\mathrm{tot}}: \{0, 1, 2, 3, 4, 5\} \to \{0, 1\}$ such that

$$f(x_1, x_2, x_3) = f^{\mathrm{tot}}(x_1 + x_2 + x_3)$$
$$T_2 f(x_1, x_2, x_3, x_4, x_5) = T_2 f^{\mathrm{tot}}(x_1 + x_2 + x_3 + x_4 + x_5)$$

for all $x_i \in \{0, 1\}$ $(i = 1, 2, 3, 4, 5)$. From (6), it follows that $T_2 f^{\mathrm{tot}}$ satisfies the equations

$$T_2 f^{\mathrm{tot}}(5) = f^{\mathrm{tot}}(3) \tag{7}$$
$$T_2 f^{\mathrm{tot}}(4) = T_2 f^{\mathrm{tot}}(3) = f^{\mathrm{tot}}(2) \tag{8}$$
$$T_2 f^{\mathrm{tot}}(2) = T_2 f^{\mathrm{tot}}(1) = f^{\mathrm{tot}}(1) \tag{9}$$
$$T_2 f^{\mathrm{tot}}(0) = f^{\mathrm{tot}}(0). \tag{10}$$

These relations determine a unique function $T_2 f^{\mathrm{tot}}$. If the numerical code of rule f is

$$C_f = \sum_{n=0}^{3} 2^n f^{\mathrm{tot}}(n)$$

the numerical code of rule $T_2 f$ is

$$C_{T_2 f} = 2^0 f^{\mathrm{tot}}(0) + (2^1 + 2^2) f^{\mathrm{tot}}(1) + (2^3 + 2^4) f^{\mathrm{tot}}(2) + 2^5 f^{\mathrm{tot}}(3).$$

For instance, if $C_f = 6 = 2 + 2^2$ then $C_{T_2 f} = 2 + 2^2 + 2^3 + 2^4 = 30$.

The result obtained for $r = 1$ and $b = 2$ can be extended to all positive integral values of r and b. Given a range-r totalistic rule f such that

$$f(x_1, x_2, \ldots, x_{2r+1}) = f^{\text{tot}}(x_1 + x_2 + \cdots + x_{2r+1})$$

there exist a unique range-rb totalistic rule $T_b f$ that satisfies the equation $T_b \mathbf{f} \circ H_b = H_b \circ \mathbf{f}$. If rule $T_b f$ is totalistic, there exists a function $T_b f^{\text{tot}}$ such that

$$T_b f(x_1, x_2, \ldots, x_{2rb+1}) = T_b f^{\text{tot}}(x_1 + x_2 + \cdots + x_{2rb+1})$$

then it satisfies the following equations:

$$T_b f^{\text{tot}}(0) = f^{\text{tot}}(0) \tag{11}$$

$$T_b f^{\text{tot}}\left((2r-k)b\right) = \cdots = T_b f^{\text{tot}}\left((2r-k-1)b+1\right) = f^{\text{tot}}(2r-k) \tag{12}$$

$$T_b f^{\text{tot}}(2rb+1) = f^{\text{tot}}(2r+1) \tag{13}$$

where k in (12) takes the values $0, 1, \ldots, 2r-1$. From these relations it follows that if the numerical code of rule f is

$$C_f = \sum_{n=0}^{2r+1} 2^n f^{\text{tot}}(n) \tag{14}$$

then the numerical code of rule $T_b f$ is

$$C_{T_b f} = 2^0 f^{\text{tot}}(0) + \sum_{k=0}^{2r-1} 2^{(2r-k-1)b+1}(2^b - 1) f^{\text{tot}}(2r-k) + 2^{2rb+1} f^{\text{tot}}(2r+1). \tag{15}$$

The numerical code of rule $T_b f$ is easy to derive from the binary representation of the numerical code of rule f. If, for instance, f is the range-2 totalistic rule 50 whose binary representation is 110010, the binary representation of the numerical code of the range-4 totalistic rule $T_2 f$ is 1110000110. That is, $C_{T_2 f} = 902$.

On the set $\{T_b | b \in \mathbb{N}\}$ of all transformations, it is possible to define the following combination law:

$$T_{b_1} T_{b_2} = T_{b_1 b_2}. \tag{16}$$

This law is associative, there exists an identity, and each element has a left inverse but, in general, no right inverse.

Note that we have *not* proved that the limit set of $T_b f$ is $H_b\left(\Lambda_f\right)$, but only that this set is invariant under $T_b \mathbf{f}$. Since a limit set is a maximal invariant subset, it follows that

$$H_b\left(\Lambda_f\right) \subseteq \Lambda_{T_b f}.$$

As stated in the introduction, the equality requires another condition: the number of sequences of zeros and ones, whose length is not a multiple of b, in any configuration S_t, generated by $T_b f$ after t time steps, tends to zero as t tends to ∞.

Numerical simulations seem to indicate that this latter condition is fulfilled for all totalistic rules $T_b f$ that satisfy the equation (2). This is probably not too surprising since $T_b f$ was not only required to fulfil equation (2), but was also required to be totalistic. This strong requirement led to a unique block transform of f: $T_b f$.

Consider, for instance, range-1 totalistic rule 6, whose spatio-temporal pattern is represented in figure 2(a). Its limit set is a regular language. An infinite word in this language, whose grammar is represented in figure 1(a), is a concatenation of strings of zeros and ones of even lengths. The transform of rule 6 for $b = 2$ is range-2 totalistic rule 30. Its limit set is a regular language whose grammar is shown in figure 1(b) (Boccara *et al* 1991a). If we look carefully at figure 2(b), which represents the spatio-temporal pattern generated by the evolution according to the range-2 totalistic rule 30, we observe a clear tendency to form rapidly blocks of zeros and ones whose lengths are a multiple of 4. This feature is even more evident in figure 2(c). After a contraction by a factor two in the space direction, each block of length two is represented by a unit square, and block formation explains why the resulting pattern looks similar to the pattern corresponding to $b = 1$ shown in figure 2(a). More generally, a configuration belonging to the limit set of range-r totalistic rule $2^{2r+1} - 2$, defined by

$$f_{2^{2r+1}-2}(x_1, x_2, \ldots, x_{2r+1}) = \begin{cases} 1 & \text{if } 0 < x_1 + x_2 + \cdots + x_{2r+1} < 2r+1 \\ 0 & \text{otherwise} \end{cases}$$

consists of alternating sequences of zeros and ones whose lengths are multiple of $2r$ (Boccara *et al* 1991a). For example, figure 2(d) represents the spatio-temporal pattern generated by the evolution according to range-3 totalistic rule 126 (the transform of range-1 totalistic rule 6 for $b = 3$) with space contraction by a factor three.

Of course, after a finite number of time steps, block formation is not perfect. In many cases, after a few tens of time steps, the only remaining 'defects' with respect to block formation are simple to describe. This is particularly true for the previous examples because their limit sets are easy to characterize (Boccara *et al* 1991a). A configuration belonging to the attractor of range-1 totalistic rule 6 is a concatenation of pairs of zeros and pairs of ones distributed at random with equal probabilities. After a few time steps, the evolution towards the attractor occurs through the elimination of defects or 'kinks' (Grassberger 1983) of only one type corresponding to odd sequences of zeros or ones (figure 2(a)). As shown by Grassberger, these defects exhibit a diffusive motion. When two defects meet they annihilate. Their number decreases as $t^{-1/2}$ as a function of time t. For range-r totalistic rule $2^{2r+1} - 2$, any sequence of zeros or ones, whose length ℓ is not a multiple of $2r$, contains a defect. A defect is characterized by an integer d ($0 < d < 2r$) equal to ℓ mod $2r$. Defects d_1 and d_2 combine according to the law (figures 2(c) and 2(d))

$$d = d_1 + d_2 \mod 2r.$$

Endowed with the above combination law, the set of all possible defects and the 'null-defect' $d = 0$ (which represents the absence of any defect) is isomorphic to the cyclic group $\mathbb{Z}/2r$.

As a trivial generalization to rule 18 studied by Grassberger (1983), these defects exhibit a diffusive motion, and their number decreases as $t^{-1/2}$ as a function of time. Hence, equation (2) with the property that the number of defects tends to 0 as $t \to \infty$ implies that $T_b f$ is a block transform of f in the sense of equation (1).

In most cases, the defects are not so easy to characterize and to follow in the spatio-temporal patterns because the complexity of the attractor is higher. In many cases, however, we have observed that the 'average number of defects' n_d defined as

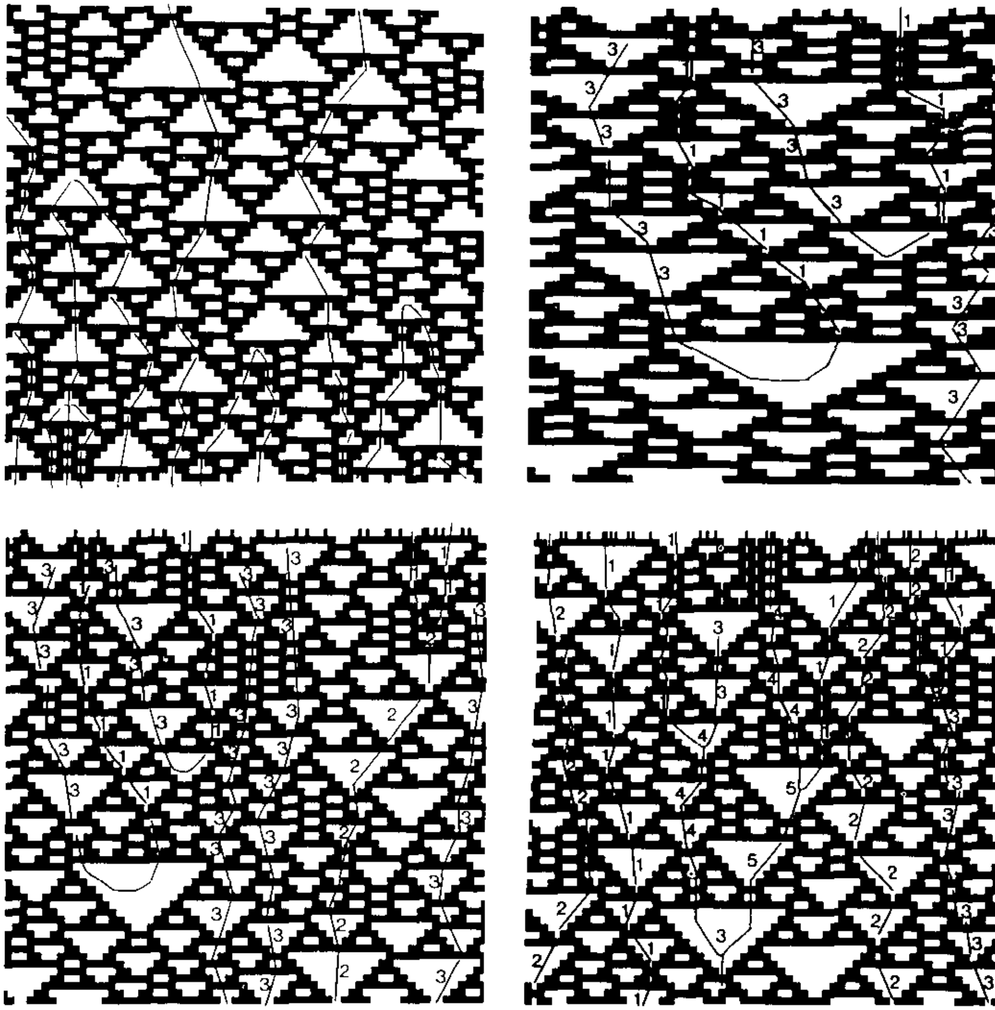

Figure 2. Spatio-temporal patterns generated by the evolution according to (a) (top left) range-1 totalistic rule 6, (b) (top right) range-2 totalistic rule 30 with no space contraction, (c) (bottom left) range-2 totalistic rule 30 with space contraction by a factor $b = 2$, (d) (bottom right) range-3 totalistic rule 126 with space contraction by a factor $b = 3$. Initial configurations are random. Numbers in (b), (c) and (d) refer to defects; they combine between themselves according to the law: $d = d_1 + d_2 \bmod 2r$. In all spatio-temporal patterns, time is oriented downward.

the sum of all sequences of zeros and ones, whose length is not a multiple of b, divided by the number of sites, decreases as $t^{-1/2}$. This is sufficient to prove that T_b is a block transformation. We give two examples. Figures 3(a) and (b) show the spatio-temporal patterns generated by the evolution of, respectively, range-1 totalistic rule 2 and its block transform for $b = 2$, which is range-2 totalistic rule 6, with a space contraction by a factor two. Figure 3(c) shows the average number of defects n_d as a function of time for the latter. Figures 4(a)–(c) use the same conventions for range-2 totalistic rule 50 and its block transform for $b = 2$, which is range-4 totalistic rule 902.

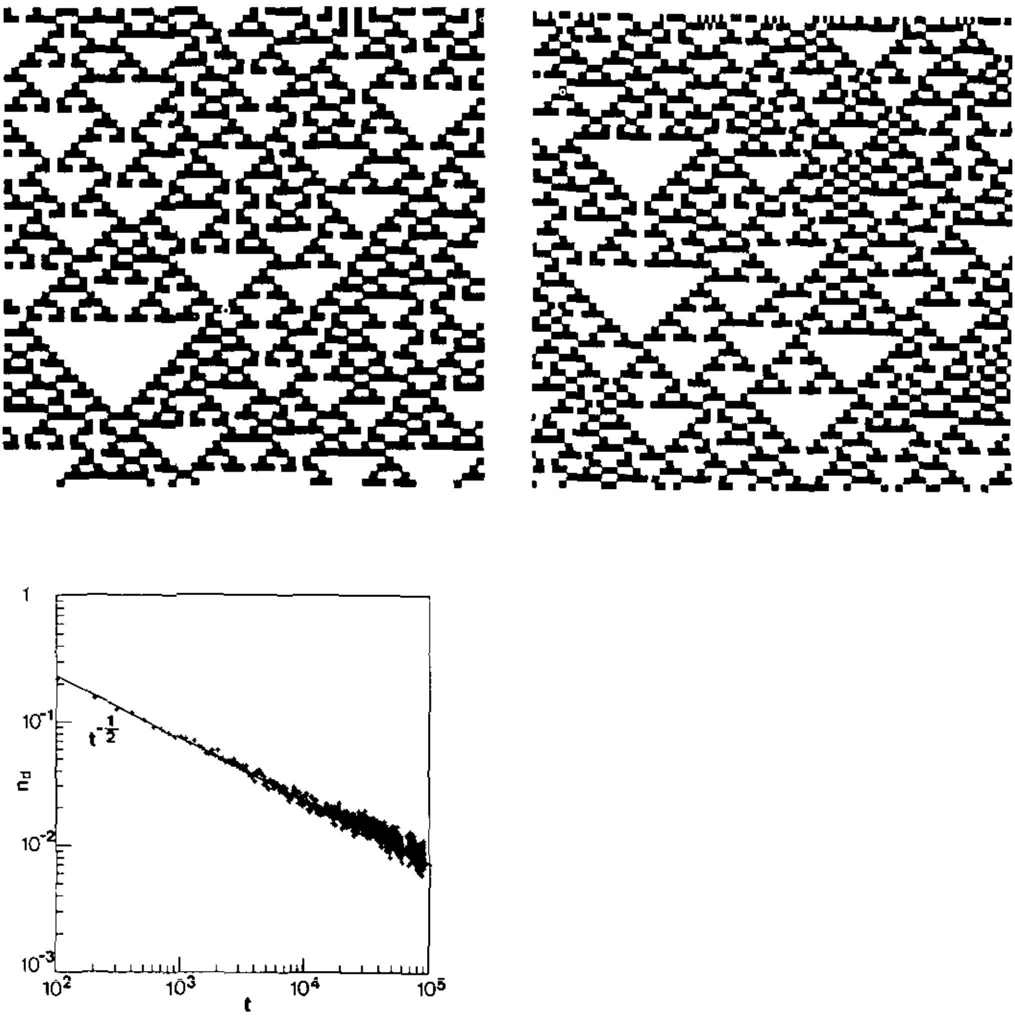

Figure 3. Spatio-temporal patterns generated by the evolution according to (a) (top left) range-1 totalistic rule 2, (b) (top right) range-2 totalistic rule 6 with space contraction by a factor two. Initial configurations are randomly generated. (c) (bottom left) Asymptotic behaviour of the average number of defects n_{d}; the size of the lattice is 10^4; periodic boundary conditions were used.

3. More general rules

In the previous section, given a range-r totalistic rule f, we proved that, for all positive integers b, there exists a unique range-rb totalistic rule $T_b f$, which satisfies the equation $T_b\mathbf{f} \circ H_b = H_b \circ \mathbf{f}$. If we do not require rule $T_b f$ to be totalistic, it is possible to establish the following more general result.

Given a range-r rule f, for all positive integers b, there exist $2^{2^{2rb+1}-2^{2r+1}b+2}$ range-rb rules that satisfy the equation $T_b\mathbf{f} \circ H_b = H_b \circ \mathbf{f}$. If f is not totalistic, none of these rules is totalistic.

Consider first the case $r = 1$ and $b = 2$. In order to fulfil the relation $T_b\mathbf{f} \circ H_b =$

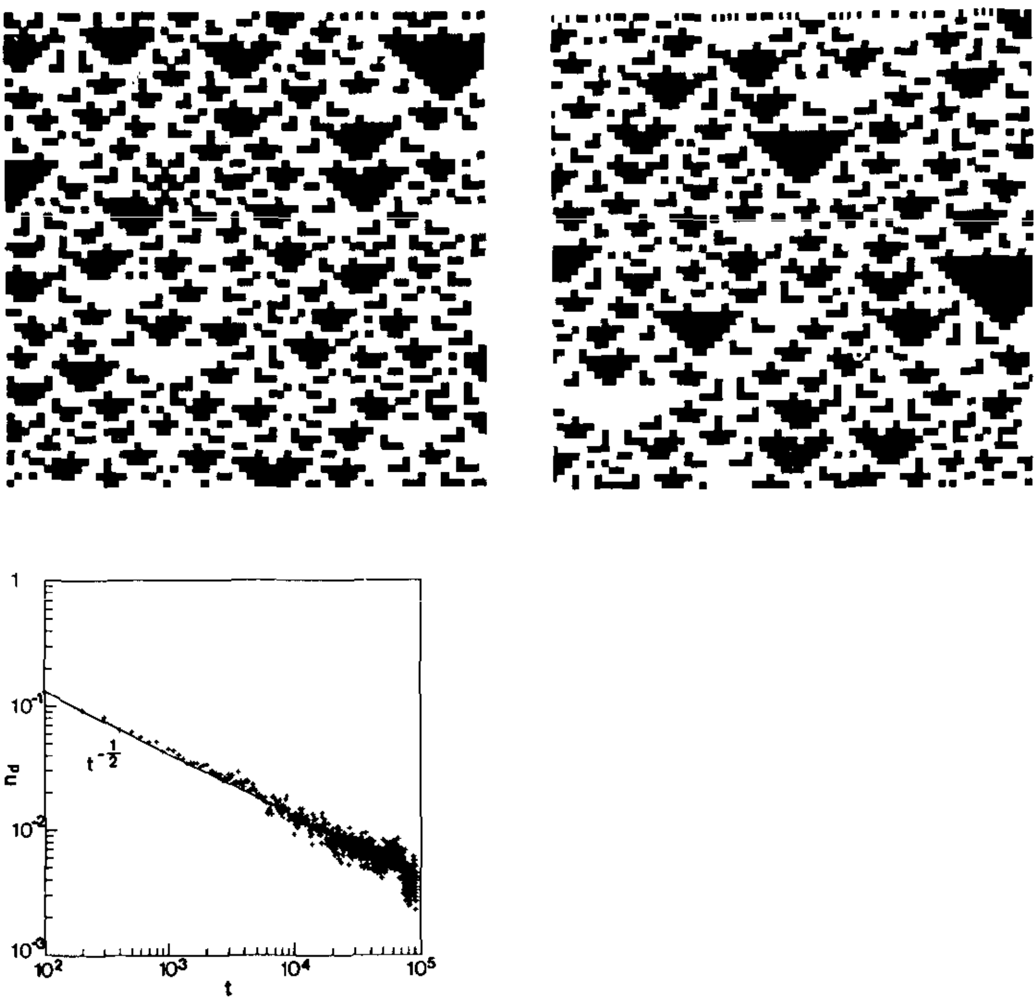

Figure 4. Spatio-temporal patterns generated by the evolution according to (*a*) (top left) range-2 totalistic rule 50, (*b*) (top right) range-4 totalistic rule 902 with space contraction by a factor two. Initial configurations are randomly generated. (*c*) (bottom left) Asymptotic behaviour of the average number of defects n_{d}; the size of the lattice is 10^4.

$H_b \circ \mathbf{f}$, the range-2 rule $T_2 f$ must satisfy (6). That is, for all $(x_1, x_2, x_3) \in \{0,1\}^3$

$$T_2 f(x_1, x_1, x_2, x_2, x_3) = T_2 f(x_1, x_2, x_2, x_3, x_3) = f(x_1, x_2, x_3). \qquad (17)$$

These relations do not determine a unique function

$$(x_1, x_2, x_3, x_4, x_5) \mapsto T_2 f(x_1, x_2, x_3, x_4, x_5).$$

Only the images of 14 five-tuples $(x_1, x_2, x_3, x_4, x_5)$ are determined by (17). The images of the remaining 18 five-tuples are arbitrary. There exist, therefore, 2^{18} rules $T_2 f$ that fulfil the relation $T_b \mathbf{f} \circ H_b = H_b \circ \mathbf{f}$. 2^{12} of them are legal. If we want $T_2 f$ to be totalistic, then there must exist a function $F: \{0,1,2,3,4,5\} \to \{0,1\}$ such that

$$F(x_1 + x_2 + x_3 + x_4 + x_5) = T_2 f(x_1, x_2, x_3, x_4, x_5)$$

with

$$
\begin{array}{rcrcl}
 & & F(0) & = & f(0,0,0) \\
F(1) & = & F(2) & = & f(0,0,1) \\
 & & F(2) & = & f(0,1,0) \\
F(3) & = & F(4) & = & f(0,1,1) \\
F(1) & = & F(2) & = & f(1,0,0) \\
 & & F(3) & = & f(1,0,1) \\
F(3) & = & F(4) & = & f(1,1,0) \\
 & & F(5) & = & f(1,1,1).
\end{array}
\tag{18}
$$

These relations show that F exists if and only if

$$f(0,0,1) = f(0,1,0) = f(1,0,0)$$

$$f(0,1,1) = f(1,0,1) = f(1,1,0).$$

That is, rule $T_2 f$ is totalistic if, and only if, f is totalistic.

In order to test block formation we have studied the $2^{12} = 4096$ range-2 legal rules corresponding to range-1 rule 18 for $b = 2$ on a 8×10^3 site lattice. For each of these rules the average number of defects n_{d} with respect to block formation was measured after 10^2, 10^3 and 10^4 time steps. The results are summarized in figure 5. The full curve represents the number $n(x)$ of rules having, after 10^3 iterations, a number of defects n_{d} smaller than x. The chain curve represents the number $n'(x)$ of rules having, after 10^3 iterations, a number of defects n_{d} such that $x - \Delta x < n_{\mathrm{d}} < x + \Delta x$, where the finite step Δx has been arbitrarily chosen equal to 0.0005. In the limit $\Delta x \to 0$, $n'(x)$ is the derivative of $n(x)$ and represents the 'density of rules' having a number of defects equal to x. Three regions can be roughly distinguished.

(i) Rules corresponding to region (A) have a small number of defects: $x < 0.03$ after 10^3 time steps. Measurements of n_{d} at 10^2, 10^3 and 10^4 time steps are compatible with a decrease of the number of defects as $t^{-1/2}$ as observed in figures 3(c) and 4(c). Such rules (about 25% of these 4096 rules) are block transforms of range-1 rule 18 with $b = 2$. In many cases the defects are very simple and obey combination laws as described in the previous section. For example, figures 6(a) and (b) show, respectively, the spatio-temporal pattern for range-1 rule 18 and range-2 rule 87 493 422. The defects are frequently more complicated, but it is still possible to describe their interactions. For example, figure 7(a) shows the spatio-temporal pattern generated by the evolution of range-2 rule 769 861 902, in which some 'extended' defects appear (Boccara *et al* 1991b). Their number also decreases as $t^{-1/2}$ as t tends to ∞ (figure 7(b)), although large fluctuations of n_{d} are observed, in relation with the creation and annihilation of extended defects (figure 7(a)).

(ii) Rules corresponding to region (C) generate intricate spatio-temporal patterns that look very different from the spatio-temporal pattern generated by range-1 rule 18 (figure 8). The number of defects n_{d} is larger than 15%. This represents about 60% of all 4096 rules.

(iii) For the rules corresponding to the intermediate region (B) there is a certain tendency to block formation in some region of the spatio-temporal pattern, but the defects are intricately interwoven and they do not seem to disappear (figure 9), (their number is roughly the same after 10^2, 10^3 and 10^4 iterations).

These results seem to be valid for most block transformations of class-3 range-1 legal rules for $b = 2$. However, in the case of range-1 rule 54 a rather peculiar

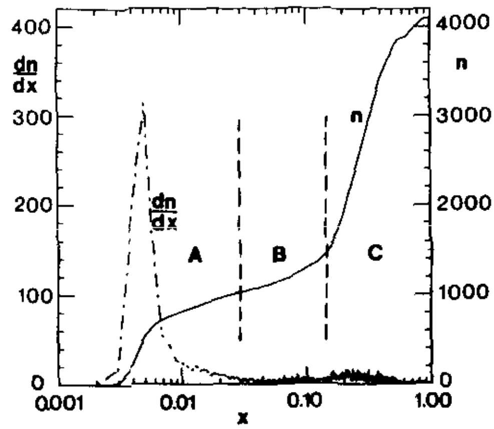

Figure 5. A study of the 4096 legal range-2 rules satisfying the condition $T_b\mathbf{f} \circ H_b = H_b \circ \mathbf{f}$ to be block transforms with $b = 2$ of range-1 rule 18. The full curve represents the number of rules $n(x)$ having, after 1000 iterations a number of defects n_d, with respect to block-formation, smaller than x. The chain curve represents the derivative of $n(x)$ or, more precisely, the number of rules having after 1000 iterations a number of defects n_d such that $x - 0.0005 < n_d < x + 0.0005$. Three regions can be roughly distinguished. Rules corresponding to region (A) have a small number of defects, $x < 0.03$, decreasing roughly as $t^{-1/2}$ as a function of time; they are block transforms of range-1 rule 18 (figures 6 and 7). Rules corresponding to region (C) have intricate patterns, very different from the pattern generated by the range-1 rule 18 (figure 8). Rules corresponding to the intermediate region (B) show a certain tendency to block formation in some region of the spatio-temporal pattern, but the defects are intricately interwoven and they do not seem to disappear—their number is roughly constant from 10^2 to 10^4 iterations (figure 9).

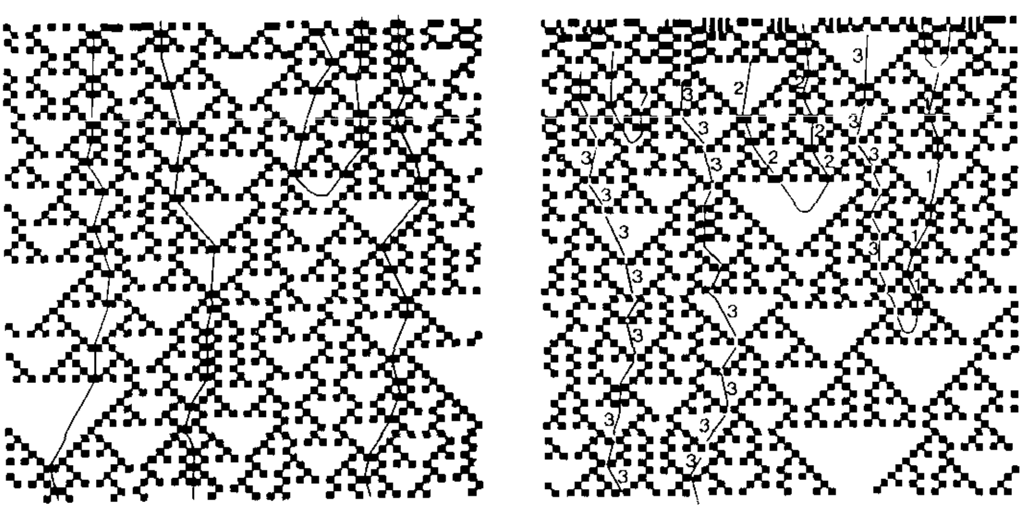

Figure 6. Spatio-temporal patterns generated by the evolution according to (*a*) (top left) range-1 rule 18, (*b*) (top right) range-2 rule 87 493 422 with space contraction by a factor two. Initial configurations are randomly generated. Numbers refer to defects; they combine according to the law: $d = d_1 + d_2 \bmod 4$.

behaviour is observed. Configurations generated by the evolution according to rule 54

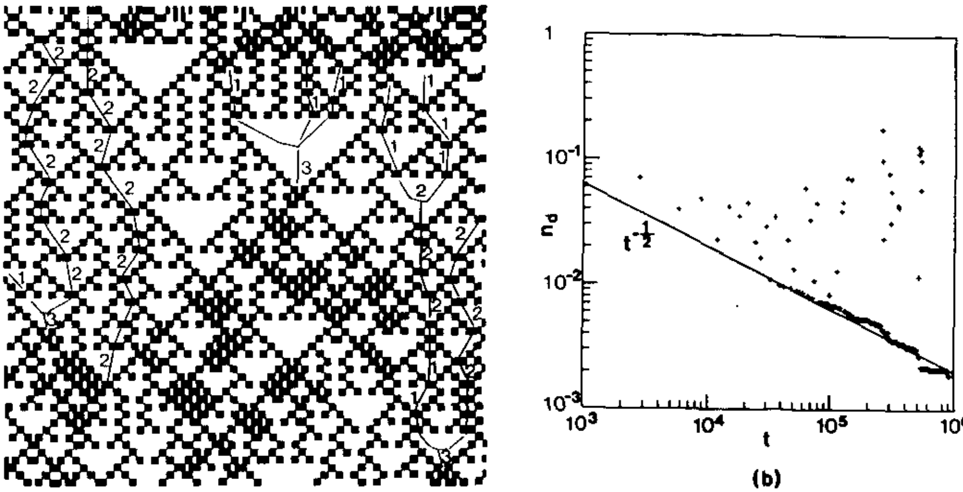

Figure 7. (*a*) (top left) Spatio-temporal patterns generated by the evolution according to range-2 rule 769 861 902 with space contraction by a factor two. Initial configurations are randomly generated. Numbers refer to defects. Defects 1 and 2 are similar to those observed in the previous figure. Defect 3 extends on both sides with velocity 1; it can be annihilated through the interaction with other defects. (*b*) (top right) Asymptotic behaviour of the average number of defects n_d. n_d is defined here as the average number of sequences of zeros and ones whose length is not equal to 2 modulo 4. The size of the lattice is 10^4; since type 3 defects may have large extension, values of n_d were averaged over 500 time steps.

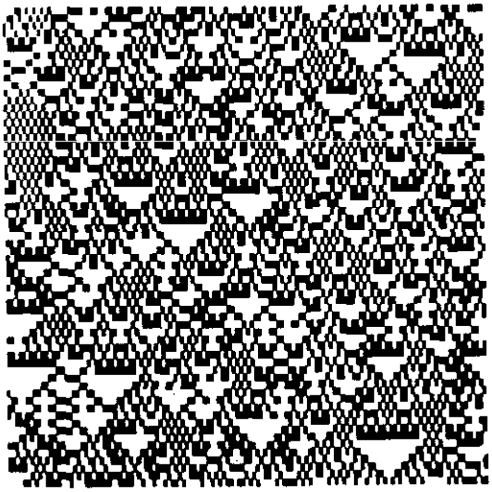

Figure 8. Spatio-temporal pattern generated by the evolution according to range-2 rule 700 802 830 with space contraction by a factor two, after 10^3 iterations. Although the equation $T_b\mathbf{f} \circ H_b = H_b \circ \mathbf{f}$ is fulfilled, there is no block formation.

may be interpreted in terms of particle-like structures evolving in a regular background (Boccara *et al* 1991b). Starting from a random initial configuration, a spatio-temporal pattern generated by this rule is shown in figure 10(*a*). The background is periodic

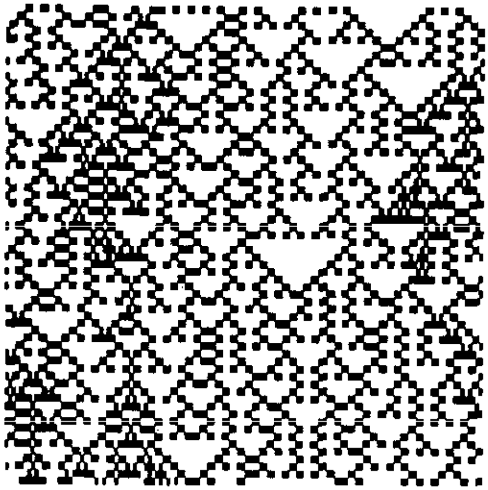

Figure 9. Spatio-temporal pattern generated by the evolution according to range-2 rule 219 500 334 with space contraction by a factor two, after 10^3 iterations. The average number of defects $n_d \approx 7\%$ does not decrease appreciably from 10^2 to 10^4 iterations.

in space and time, both periods being equal to 4. Three types of particles may be distinguished. Two of them are non-propagating and periodic in time. Their periods are equal to 4. They are denoted, respectively, by g_e and g_o (g stands for 'gutter') according to whether they consist of sequences of zeros of even or odd lengths. There exists also a propagating particle w (w stands for 'wall'). This particle may propagate to the right or to the left. Its velocity is equal to 1.

For many range-2 legal rules which are block transforms of range-1 rule 54, after a short transient time, all defects with respect to block formation are 'trapped' in the gutters g_e as shown in figure 10(b). Hence, the motion of these defects is then directly related to the motion of the particles g_o, g_e and w which are not diffusive. The number of defects decreases much more slowly than $t^{-1/2}$. Actually, it decreases approximately as $t^{-\gamma}$ with $\gamma \ll 1/2$ (Boccara *et al* 1991b).

These results may be extended to all positive integral values of r and b. Given a range-r rule f and a positive integer b, relations similar to (17) determine the images of only $2^{2r+1}b - 2$ '$(2rb+1)$-tuples' among the 2^{2rb+1}. Therefore, there exist $2^{2^{2rb+1}-2^{2r+1}b+2}$ range-rb rules that fulfil the condition $T_b\mathbf{f} \circ H_b = H_b \circ \mathbf{f}$. Relations similar to (18) show that if f is not totalistic none of these rules is totalistic.

4. Block transformations of arbitrary range

Given a range-r rule f and a block size b, we have found that there exist range-rb rules that satisfy the relation $T_b\mathbf{f} \circ H_b = H_b \circ \mathbf{f}$. In fact, for any positive integer $R \geq rb$, there exist range-R rules that fulfil the condition $T_b\mathbf{f} \circ H_b = H_b \circ \mathbf{f}$.

We shall establish this result for $r = 1$, $b = 2$ and $R = 3$. The generalization is straightforward.

In order to fulfil the condition $T_b\mathbf{f} \circ H_b = H_b \circ \mathbf{f}$, the range-3 rule $T_2 f$ must satisfy the following relations, which are the analogue of (6). That is, for all

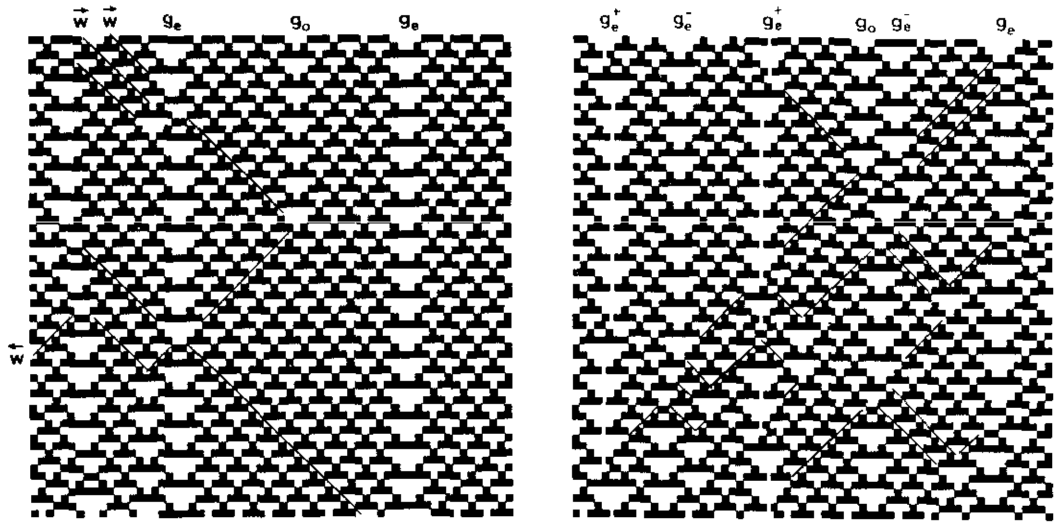

Figure 10. Spatio-temporal patterns generated by the evolution according to (a) (top left) range-1 rule 54; it can be described through 'particles' as 'walls' (w), and even (g_e) and odd (g_o) 'gutters', interacting on a periodic background. (b) (top right) range-2 rule 118 167 626 with space contraction by a factor two, after 90 time steps. Initial configuration were randomly generated. After a few tens of time steps, the only remaining defects are g_e^+ and g_e^- which can be considered to be even 'gutters' which have trapped complementary defects.

$(x_1, x_2, x_3, x_4, x_5) \in \{0,1\}^5$

$$T_2 f(x_1, x_2, x_2, x_3, x_3, x_4, x_4) = T_2 f(x_2, x_2, x_3, x_3, x_4, x_4, x_5) = f(x_2, x_3, x_4). \qquad (19)$$

These relations do not determine a unique function

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto T_2 f(x_1, x_2, x_3, x_4, x_5, x_6, x_7).$$

Only the images of 30 'seven-tuples' $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ are determined by (19). The images of the remaining 98 'seven-tuples' are arbitrary. There exist, therefore, 2^{98} rules $T_2 f$ that fulfil the condition $T_b \mathbf{f} \circ H_b = H_b \circ \mathbf{f}$. If we want $T_2 f$ to be totalistic, then there must exist a function $F\colon \{0,1,2,3,4,5,6,7\} \to \{0,1\}$ such that

$$F(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7) = T_2 f(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$$

with

$$\begin{array}{lllll}
F(0) & = & F(1) & = & f(0,0,0) \\
F(2) & = & F(3) & = & f(0,0,1) \\
F(2) & = & F(3) & = & f(0,1,0) \\
F(4) & = & F(5) & = & f(0,1,1) \\
F(2) & = & F(3) & = & f(1,0,0) \\
F(4) & = & F(5) & = & f(1,0,1) \\
F(4) & = & F(5) & = & f(1,1,0) \\
F(6) & = & F(7) & = & f(1,1,1).
\end{array} \qquad (20)$$

These relations show that F exists if and only if

$$f(0,0,1) = f(0,1,0) = f(1,0,0)$$
$$f(0,1,1) = f(1,0,1) = f(1,1,0).$$

That is, rule $T_2 f$ is totalistic if and only if f is totalistic.

If f is totalistic, relations (20) show that there exist a unique range-3 totalistic rule $T_2 f$. If the numerical code of rule f is

$$C_f = \sum_{n=0}^{3} 2^n f^{\mathrm{tot}}(n)$$

where f^{tot} is such that, for all $(x_1, x_2, x_3) \in \{0,1\}^3$

$$f(x_1, x_2, x_3) = f^{\mathrm{tot}}(x_1 + x_2 + x_3)$$

the numerical code of rule $T_2 f$ is

$$C_{T_2 f} = (2^0 + 2^1) f^{\mathrm{tot}}(0) + (2^2 + 2^3) f^{\mathrm{tot}}(1) + (2^4 + 2^5) f^{\mathrm{tot}}(2) + (2^6 + 2^7) f^{\mathrm{tot}}(3).$$

For instance, if $C_f = 6 = 2 + 2^2$ then $C_{T_2 f} = 2^2 + 2^3 + 2^4 + 2^5 = 60$.

The generalization of all these results is not very difficult: given a range-r rule f and an integer $b > 1$, for all positive integers $R \geq rb$, there exist range-R rules that satisfy the equation $T_b \mathbf{f} \circ H_b = H_b \circ \mathbf{f}$. If f is not totalistic, none of these rules is totalistic. If f is totalistic, then there exists a unique totalistic range-R rule $T_b f$ that fulfils the condition $T_b \mathbf{f} \circ H_b = H_b \circ \mathbf{f}$.

As in the previous section it is difficult to find further criteria to identify, among all the rules obeying the equation $T_b \mathbf{f} \circ H_b = H_b \circ \mathbf{f}$, the rules that are really block transforms. By imposing more constraints on $T_b f$ it has, however, been possible to define a particular non-totalistic block transform. This particular block transformation has been defined and studied in a recent paper by Boccara (1989). It is characterized by a positive *odd* integer b. In order to build up this transformation T_b, consider the set

$$\{s(j-(b-1)/2, t), s(j-(b-3)/2, t), \ldots, s(j+(b-1)/2, t)\}$$

which constitutes a block of site variables at time t. Its length is b, and it is centred at j. To a block centred at j associate $B(j,t)$, called a block variable, such that (majority rule)

$$B(j,t) = \begin{cases} 1 & \text{if } S_b(j,t) > b/2 \\ 0 & \text{if } S_b(j,t) < b/2 \end{cases}$$

where

$$S_b(j,t) = s(j-(b-1)/2, t) + s(j-(b-3)/2, t) + \cdots + s(j+(b-1)/2, t).$$

With a range-r rule f associate a range-$rb + (b-1)/2$ rule $T_b f$, i.e. a rule involving a $(2r+1)b$ neighbourhood, defined as follows. Divide the $(2r+1)b$ site variables in

$2r+1$ blocks of length b. For each block determine the value of the corresponding block variable. Then the value at time $t+1$ of site variable $s(i,t+1)$ given by rule $T_b f$ is, by definition, given by rule f applied to the block variables centred at $i+lb$, where $l=-r,-(r-1),\ldots,r$, at time t. For example, the transform of range-1 rule 18 for $b=3$ is the range-4 rule $T_3 f_{18}$ such that

$$T_3 f(x_1, x_2, \ldots, x_9) = 1$$

if and only if

$$x_1+x_2+x_3 < \tfrac{3}{2} \qquad x_4+x_5+x_6 < \tfrac{3}{2} \qquad x_7+x_8+x_9 > \tfrac{3}{2}$$

or

$$x_1+x_2+x_3 > \tfrac{3}{2} \qquad x_4+x_5+x_6 < \tfrac{3}{2} \qquad x_7+x_8+x_9 < \tfrac{3}{2}.$$

5. Conclusion

Given a rule f, with limit set Λ_f and a positive integer b, the rule $T_b f$ is a block transform of f if its limit set $\Lambda_{T_b f}$ satisfies the relation (1)

$$\Lambda_{T_b f} = H_b\left(\Lambda_f\right)$$

where the mapping $H_b\colon \mathcal{S} \to \mathcal{S}$ is a homomorphism defined by

$$(\forall \cdots xyz \cdots \in \mathcal{S}) \qquad H_b(\cdots xyz \cdots) = \cdots x^b y^b z^b \cdots.$$

x^b stands for x repeated b times; for instance, $x^3 = xxx$.

The condition (2)

$$T_b\mathbf{f} \circ H_b = H_b \circ \mathbf{f}$$

to be satisfied for a block transform $T_b f$ to exist does not completely determine $T_b f$ if f is not totalistic. If f is totalistic, there exists a unique $T_b f$, and it seems that in this case condition (2) is sufficient. If f is not totalistic, condition (2) is not sufficient. Among the very large number of rules satisfying equation (2), only a fraction seems to be block transforms and none of them is totalistic. The range of a block transform $T_b f$ may be any integer greater than or equal to rb.

The evolution according to a block transform $T_b f$ towards its limit set has been discussed in terms of annihilating defects. These defects are often simply related to the defects characterizing the evolution according to rule f.

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